WNE Linear Algebra
Final Exam
Series A

2 February 2023

Questions

Question 1.
Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an endomorphism given by the formula

$$\varphi((x_1, x_2)) = (3x_1 - x_2, tx_1 + 2x_2).$$

For which $t \in \mathbb{R}$ is the vector $v = (1, 1)$ an eigenvector of φ? Find the corresponding eigenvalue.

Solution 1.

$$\varphi((1, 1)) = (2, t + 2) = \lambda(1, 1),$$

if and only if $\lambda = 2$ and $t + 2 = 2$, i.e. $t = 0$.

Question 2.
Let $A \in M(n \times n; \mathbb{R})$, $\det A \neq 0$ be a diagonalizable matrix. If $C^{-1}AC = D$ is a diagonal matrix for some invertible matrix $C \in M(n \times n; \mathbb{R})$, does it follow that columns of C^{-1} are eigenvectors of the matrix A^{-1}?

Solution 2.
If $C^{-1}AC = D$ and A is invertible then $CA^{-1}C = D^{-1}$ which implies that columns of C are eigenvectors of A^{-1}. The answer is no, it does not. For example, if $(1, 2), (1, 3)$ are eigenvectors corresponding to different eigenvalues then $C = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ and $C^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$, but subspaces

$$\mathrm{lin}((1, 2)), \quad \mathrm{lin}((1, 3)), \quad \mathrm{lin}((3, -2)), \quad \mathrm{lin}((-1, 1)),$$

are pairwise different. That is, if $(1, 2)$ and $(1, 3)$ are eigenvectors of matrix A then $(3, -2)$ and $(-1, 1)$ are not eigenvectors of matrix A^{-1}.

Question 3.
If $A \in M(2 \times 2; \mathbb{R})$ is an antisymmetric matrix, i.e. $A^\intercal = -A$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the unit matrix, does it follow that matrix $A + I$ is invertible?

Solution 3.
Matrix A is antisymmetric if and only if $A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$, for some $a \in \mathbb{R}$. Hence

$$\det(A + I) = 1 + a^2 > 0,$$

which implies that $A + I$ is invertible.

Question 4.
Matrix $P = \begin{bmatrix} \frac{1}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{9}{10} \end{bmatrix}$ is a matrix of an orthogonal projection onto some subspace $V \subset \mathbb{R}^2$. Find an orthonormal basis of $V^\perp$.

Solution 4.
The subspace V is equal to image of P, i.e. $\text{lin}((1,3))$ (spanned by columns of P). The subspace $V^{\perp}$ is equal to the kernel of P, which is orthogonal to V, i.e. $\text{lin}((3,-1))$. An orthonormal basis of $V^{\perp}$ is, for example, $\frac{1}{\sqrt{10}}(3,-1)$.

Question 5.
A system of linear equations in two variables has two solutions $(1,3)$ and $(1,5)$. Give an example of a third solution different from the previous ones.

Solution 5.
The set of all solutions of a system of linear equations is an affine subspace hence it is closed under affine combinations. Another solutions is for example

$$\frac{1}{2}(1,3) + \frac{1}{2}(1,5) = (1,4).$$

In fact, for any $b \in \mathbb{R}$ vector $(1,b)$ is a solution of that system.